

Violation of the Robertson-Schrödinger uncertainty principle and non-commutative quantum mechanics

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We show that a possible violation of the Robertson-Schrödinger uncertainty principle may signal the existence of a deformation of the Heisenberg-Weyl algebra. More precisely, we prove that any Gaussian in phase-space (even if it violates the Robertson-Schrödinger uncertainty principle) is always a quantum state of an appropriate non-commutative extension of quantum mechanics. Conversely, all canonical non-commutative extensions of quantum mechanics display states that violate the Robertson-Schrödinger uncertainty principle.

1. Since its inception, quantum mechanics was formulated in terms of Hilbert spaces and self-adjoint operators acting therein. In this context Heisenberg's uncertainty relations become a straightforward consequence of the non-commutativity of the fundamental operators of position \hat{Q} and momentum \hat{P} and the Cauchy-Schwarz inequality. For a n -dimensional system, the Heisenberg-Weyl (HW) algebra reads:

$$[\hat{Q}_i, \hat{P}_j] = i\delta_{i,j}, \quad i, j = 1, \dots, n, \quad (1)$$

and all remaining commutators vanish. Since our results are much more general than simple rescalings of Planck's constant, we have set $\hbar = 1$ for the

remainder of this work and assumed that position and momentum have the same units. If we define the phase-space variable $\hat{Z} = (\hat{Q}, \hat{P})$, we have in more compact notation

$$[\hat{Z}_i, \hat{Z}_j] = iJ_{ij}, \quad i, j = 1, \dots, 2n, \quad (2)$$

where $\mathbf{J} = (J_{ij})$ is the standard symplectic matrix

$$\mathbf{J} = -\mathbf{J}^T = -\mathbf{J}^{-1} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}. \quad (3)$$

A simple calculation leads to the inequalities

$$\Delta_{Q_j} \cdot \Delta_{P_j} \geq \frac{1}{2}, \quad j = 1, \dots, n, \quad (4)$$

where Δ_{Q_j} and Δ_{P_j} denote the mean standard-deviations of \hat{Q}_j and \hat{P}_j with respect to an arbitrary state. The set of inequalities (4) are known as the Heisenberg-Weyl-Pauli inequalities. They are not invariant under linear symplectic transformations of the operators nor under metaplectic transformations of the states. This prompted the search for an alternative set of inequalities which are stronger than the Heisenberg inequalities and have the right symplectic covariance properties. They are known

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as the Robertson-Schrödinger uncertainty principle (RSUP) [1]:

$$\Sigma + \frac{i}{2}\mathbf{J} \geq 0. \quad (5)$$

Here Σ denotes the covariance matrix with respect to an arbitrary state and has entries $\Sigma_{ij} = \langle (\hat{Z}_i - \langle \hat{Z}_i \rangle) (\hat{Z}_j - \langle \hat{Z}_j \rangle) \rangle$ ($i, j = 1, \dots, 2n$) and $\langle \cdot \rangle$ denotes an expectation value in a chosen state. In this work we shall focus on this form of the uncertainty principle, as it implies the Heisenberg uncertainty relations, it is invariant under linear symplectic transformations and moreover it constitutes the necessary and sufficient condition for a Gaussian to be a quantum mechanical state. In fact, Gaussians are of the utmost importance, not only because they are completely determined by their covariance matrix, but also because experimentally coherent and squeezed states play an important role in quantum optics [2], quantum computation of continuous variables [3] and investigations of the quantum-classical transition [4].

Various authors have tested theoretically and experimentally the validity of these inequalities and the consequences of their violation. For instance, Popper's experiment is usually regarded as a violation of the uncertainty principle [5]. The type of arguments used by Popper [6] are similar in spirit to those of the EPR experiment [7] so that non-locality and the uncertainty principle seem to be inextricably linked. More precisely, the degree of non-locality of any theory is determined by two factors- the strength of the uncertainty principle, and the strength of a property called "steering", which determines which states can be prepared at one location given a measurement at another [8].

In this work, we show that the breakdown of the uncertainty principle may hint that the sub-atomic world is described not by standard quantum mechanics but by a non-commutative extension [9, 10] of it. This extension is obtained by replacing HW algebra (2) by a *deformed* algebra. This leads to an extra non-commutativity between the configuration and momentum variables. The most commonly used deformed algebra reads:

$$[\hat{\Xi}_i, \hat{\Xi}_j] = i\Omega_{ij}, \quad i, j = 1, \dots, 2n, \quad (6)$$

where the matrix $\Omega = (\Omega_{ij})$ is given by:

$$\Omega = \begin{pmatrix} \Theta & \mathbf{I} \\ -\mathbf{I} & \Upsilon \end{pmatrix} \quad (7)$$

Here $\Theta = (\theta_{ij})$ and $\Upsilon = (\eta_{ij})$ are real constant skew-symmetric $n \times n$ matrices measuring the strengths of the position-position and momentum-momentum non-commutativities, respectively.

Non-commutative quantum mechanics (NCQM) is usually regarded as a non-relativistic one-particle sector of the very discussed non-commutative quantum field theories [11], which emerge in the context of string theory and quantum gravity [12].

Deformations of the HW algebra have also interesting implications for quantum cosmology and this new structure leads to the thermodynamic stability of black holes and a possible regularization of the black hole singularities [13].

In this letter, we address the properties of states of NCQM. Various features of noncommutative quantum states for the algebra (6)-(7) have already been derived [9]. Here we are concerned with the possibility of using Gaussian states to identify the correct algebraic structure for quantum mechanics. More precisely, we show that: (i) NCQM displays states which are not states in standard quantum mechanics (as they violate the RSUP); (ii) conversely, that any Gaussian state (even if it is not a state in standard quantum mechanics) is nevertheless a state of some NC quantum theory; (iii) this NC theory is not unique and one can always find a NCQM for which the Gaussian is a quantum pure state. We stress that all these results are true independently of the value of Planck's constant and we set $\hbar = 1$ for the entire paper.

Our proof takes place in the context of the Weyl-Wigner formulation or deformation quantization framework, where position and momentum variables appear on equal footing. As shown in [9], it is only in this formulation that one can tell whether states correspond to a quantization of the HW algebra or some of its deformation.

2. A symplectic form on a real vector space V is a bilinear map $\omega : V \times V \rightarrow \mathbb{R}$, which is skew-symmetric ($\omega(v, v') = -\omega(v', v)$ for all $v, v' \in V$) and non-degenerate ($\omega(v, v') = 0$ for all $v' \in V$ implies $v = 0$). The archetypal symplectic vector space is $V = \mathbb{R}^{2n} = \mathbb{R}_x^n \times \mathbb{R}_p^n$ endowed with the standard symplectic form:

$$\sigma(z, z') = z \cdot \mathbf{J} z' = p \cdot x' - x \cdot p', \quad (8)$$

where \mathbf{J} is the standard symplectic matrix and $z = (x, p)$, $z' = (x', p') \in \mathbb{R}^{2n}$.

Now, let

$$\omega(z, z') = z \cdot \Omega^{-1} z' \quad (9)$$

be another arbitrary symplectic form on \mathbb{R}^{2n} . Here Ω is some real, anti-symmetric, non-singular, $2n \times 2n$ matrix (not necessarily (7)). A well known theorem in symplectic geometry [14] states that all symplectic vector spaces of equal dimension are symplectically equivalent. In other words, there exists a real non-singular $2n \times 2n$ matrix \mathbf{S} such that

$\omega(\mathbf{S}z, \mathbf{S}z') = \sigma(z, z')$ for all $z, z' \in \mathbb{R}^{2n}$, or matrix-wise

$$\mathbf{\Omega} = \mathbf{SJS}^T. \quad (10)$$

Obviously, the matrix \mathbf{S} is not unique. Indeed, let \mathbf{P} denote a symplectic matrix ($\mathbf{P} \in Sp(2n; \sigma)$), that is $\mathbf{PJP}^T = \mathbf{J}$. Then the matrix \mathbf{SP} also satisfies (10). We shall call the set of all matrices which satisfy (10) the set of Darboux matrices associated with ω and denote it by $\mathcal{D}(2n; \omega)$.

Conversely, given any real, non-singular, $2n \times 2n$ matrix \mathbf{S} , let $\mathbf{\Omega}$ be defined by (10). Then ω given by (9) is a symplectic form on \mathbb{R}^{2n} . This simple observation will be the crux of our main result.

A. Quantization on the standard symplectic space. To quantize a system on the standard symplectic space, one resorts to the HW operators

$$\widehat{U}_\sigma(z) = e^{i\sigma(z, \widehat{Z})} = e^{i(p \cdot \widehat{Q} - x \cdot \widehat{P})} \quad (11)$$

with $z = (x, p) \in \mathbb{R}^{2n}$, and where $\widehat{Z} = (\widehat{Q}, \widehat{P})$ denote the quantum mechanical position and momentum operators.

These operators constitute a unitary irreducible representation of the HW algebra. Indeed, they satisfy the relations

$$\begin{aligned} \widehat{U}_\sigma(z) \widehat{U}_\sigma(z') &= e^{\frac{i}{2}\sigma(z, z')} \widehat{U}_\sigma(z + z') \\ &= e^{i\sigma(z, z')} \widehat{U}_\sigma(z') \widehat{U}_\sigma(z), \end{aligned} \quad (12)$$

which can be readily obtained from the HW algebra (2) through the Baker-Campbell-Hausdorff formula.

A generic linear operator acting on the Hilbert space of the system (in this case $L^2(\mathbb{R}^n)$) can then be represented by

$$\widehat{A} = (2\pi)^{-n} \int dz \tilde{\alpha}_\sigma(\mathbf{J}^{-1}z) \widehat{U}_\sigma(z), \quad (13)$$

where $\tilde{\alpha}_\sigma$ denotes the Fourier transform of some suitable tempered distribution $\alpha_\sigma(z)$ on the phase-space, commonly known as the Weyl-symbol of the operator \widehat{A} .

Since $Tr(\widehat{U}_\sigma(z)) = (2\pi)^n \delta(z)$, we obtain from (12) that $Tr(\widehat{U}_\sigma(z) \widehat{U}_\sigma(z')) = (2\pi)^n \delta(z + z')$, and (13) can be readily inverted. Thus, $\tilde{\alpha}_\sigma(z) = Tr(\widehat{A} \widehat{U}_\sigma(\mathbf{J}^{-1}z))$, and the Weyl symbol of \widehat{A} reads:

$$\begin{aligned} \widehat{A} &\mapsto W_\sigma \widehat{A} = \alpha_\sigma(z) \\ &= (2\pi)^{-n} \int dz' Tr(\widehat{A} \widehat{U}_\sigma(z')) e^{i\sigma(z', z)} \\ &= \int dy \langle x + \frac{y}{2} | \widehat{A} | x - \frac{y}{2} \rangle e^{-ip \cdot y}. \end{aligned} \quad (14)$$

This procedure establishes a one-to-one map W_σ - called Weyl correspondence - between linear operators and the associated symbols.

The state of a quantum system is represented by a positive trace-class operator, the density matrix $\widehat{\rho}$. When applied to $\widehat{\rho}$ the Weyl correspondence yields (up to a normalization constant) the celebrated Wigner function on $(\mathbb{R}^{2n}, \sigma)$:

$$W_\sigma \widehat{\rho}(x, p) = (2\pi)^{-n} \int dy \langle x + \frac{y}{2} | \widehat{\rho} | x - \frac{y}{2} \rangle e^{-ip \cdot y}. \quad (15)$$

In particular, for a pure state $\widehat{\rho} = |\psi\rangle\langle\psi|$ for $\psi \in L^2(\mathbb{R}^n)$:

$$W_\sigma \psi(x, p) = (2\pi)^{-n} \int dy \psi(x + \frac{y}{2}) \overline{\psi(x - \frac{y}{2})} e^{-ip \cdot y}. \quad (16)$$

In general, it is very difficult to assess whether a function $F(x, p)$ in phase-space is the Wigner function of some density matrix (see e.g. [15]). A notable exception are the Gaussians

$$\mathcal{G}_{\mathbf{\Sigma}, \zeta}(z) = \frac{1}{(2\pi)^n \sqrt{\det \mathbf{\Sigma}}} \exp \left[-\frac{1}{2}(z - \zeta) \cdot \mathbf{\Sigma}^{-1}(z - \zeta) \right] \quad (17)$$

where $z = (x, p) \in \mathbb{R}^{2n}$, $\zeta \in \mathbb{R}^{2n}$ and $\mathbf{\Sigma}$ is the covariance matrix, which is a real, positive-definite $2n \times 2n$ matrix. It is a well documented fact [2, 15] that such a Gaussian is a Wigner function on $(\mathbb{R}^{2n}; \sigma)$ if and only if it satisfies the RSUP (5).

We may be more specific and determine whether the Gaussian is the Wigner function of a pure state. Indeed, it has been proven [4] that (17) is the Wigner function of a pure state if and only if there exists $\mathbf{P} \in Sp(2n; \sigma)$ such that

$$\mathbf{\Sigma} = \frac{1}{2} \mathbf{P}^T \mathbf{P}. \quad (18)$$

B. Quantization on non-standard symplectic spaces. NCQM results from quantizing the classical theory on a non-standard symplectic space $(\mathbb{R}^{2n}; \omega)$ [9, 10]. In this case, the HW algebra (2) is replaced by the modified algebra (6). The HW operators (11) become

$$\widehat{U}_\omega(\xi) = e^{i\omega(\xi, \widehat{\Xi})} = e^{i\xi \cdot \mathbf{\Omega}^{-1} \widehat{\Xi}}. \quad (19)$$

and concomitantly

$$\begin{aligned} \widehat{U}_\omega(\xi) \widehat{U}_\omega(\xi') &= e^{\frac{i}{2}\omega(\xi, \xi')} \widehat{U}_\omega(\xi + \xi') \\ &= e^{i\omega(\xi, \xi')} \widehat{U}_\omega(\xi') \widehat{U}_\omega(\xi). \end{aligned} \quad (20)$$

Since $\widehat{\Xi} = \mathbf{S}\widehat{Z}$ for some $\mathbf{S} \in \mathcal{D}(2n; \omega)$, from (10):

$$\begin{aligned} \widehat{U}_\omega(\xi) &= \widehat{U}_\sigma(\mathbf{S}^{-1}\xi), \\ Tr(\widehat{U}_\omega(\xi) \widehat{U}_\omega(\xi')) &= (2\pi)^n \sqrt{\det \mathbf{\Omega}} \delta(\xi + \xi'). \end{aligned} \quad (21)$$

Substituting (21) into (13), we obtain

$$\widehat{A} = (2\pi \sqrt{\det \mathbf{\Omega}})^{-n} \int dz' \tilde{a}_\omega(\mathbf{\Omega}^{-1}z') \widehat{U}_\omega(z'), \quad (22)$$

where (10) and $\tilde{a}_\omega(u) = \tilde{a}_\sigma(\mathbf{S}^T u)$ have been used. Hence, the Weyl symbol of \hat{A} on $(\mathbb{R}^{2n}; \omega)$ is given by

$$\begin{aligned} \hat{A} \mapsto W_\omega \hat{A} &= a_\omega(\xi) = (\sqrt{\det \Omega})^{-1} a_\sigma(\mathbf{S}^{-1} \xi) = \\ &= ((2\pi)^n \det \Omega)^{-1} \int d\xi' \operatorname{Tr} \left(\hat{A} \hat{U}_\omega(\xi') \right) e^{i\omega(\xi', \xi)}. \end{aligned} \quad (23)$$

In particular if $W_\sigma \hat{\rho}$ denotes the Wigner function of a density matrix $\hat{\rho}$ on $(\mathbb{R}^{2n}; \sigma)$, then the corresponding Wigner function on $(\mathbb{R}^{2n}; \omega)$ is given by [9]

$$W_\omega \hat{\rho}(\xi) = \frac{1}{\sqrt{\det \Omega}} W_\sigma \hat{\rho}(\mathbf{S}^{-1} \xi). \quad (24)$$

C. Main result. From (24) one can derive the counterparts of the RSUP and Littlejohn's Theorem [4] for Gaussians on $(\mathbb{R}^{2n}; \omega)$. From (5) and (24), we conclude that the Gaussian (17) is a quantum state on $(\mathbb{R}^{2n}; \omega)$ if and only if

$$\Sigma + \frac{i}{2} \Omega \geq 0. \quad (25)$$

Likewise, from (18) and (24), the Gaussian is a Wigner function on $(\mathbb{R}^{2n}; \omega)$ of a pure state if and only if there exists a matrix $\mathbf{C} \in \mathcal{D}(2n; \omega)$ such that

$$\Sigma = \frac{1}{2} \mathbf{C} \mathbf{C}^T. \quad (26)$$

We are now in condition prove our main result.

Let $\mathcal{G}_{\Sigma, \zeta}$ be a Gaussian of the form (17). Then there exists a matrix Ω associated with a symplectic form (9), such that $\mathcal{G}_{\Sigma, \zeta}$ is a quantum state of the NCQM based on the deformed Heisenberg algebra (6).

Indeed, since Σ is positive definite, there exists a real, non-singular, $2n \times 2n$ matrix \mathbf{C} for which (26) holds. Define a matrix Ω by $\Omega = \mathbf{C} \mathbf{J} \mathbf{C}^T$. Clearly, the form ω defined by (9) is a symplectic form and $\mathbf{C} \in \mathcal{D}(2n; \omega)$. According to Littlejohn's Theorem on $(\mathbb{R}^{2n}; \omega)$ (cf.(26)), then $\mathcal{G}_{\Sigma, \zeta}$ is a Wigner function on $(\mathbb{R}^{2n}; \omega)$ of a pure state. This concludes the proof.

Notice that, as a by-product of the proof, for any Gaussian, one can always find a NCQM for which the Gaussian is a pure state. In a certain sense this procedure amounts to a purification of the state.

There is a converse result of the previous one. Namely: given a NCQM we can always find quantum states which violate the standard RS uncertainty principle. A general proof of this result is beyond the scope of the present letter. Here we shall illustrate explicitly this result for NCQM in 2 dimensions with algebra (6)-(7). That is

$$\Omega = \begin{pmatrix} \theta \mathbf{E} & \mathbf{I} \\ -\mathbf{I} & \eta \mathbf{E} \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (27)$$

where $\theta, \eta > 0$ are real constants such that $\xi \frac{4}{\theta \eta} < 1$. A simple Darboux matrix is $\mathbf{C} \in \mathcal{D}(2n; \omega)$ given by

$$\mathbf{C} = \begin{pmatrix} \lambda \mathbf{I} & -\frac{\theta}{2\lambda} \mathbf{E} \\ \frac{\eta}{2\mu} \mathbf{E} & \mu \mathbf{I} \end{pmatrix} \quad (28)$$

where μ, λ are real parameters such that $2\mu\lambda = 1 + \sqrt{1 - \xi}$. By construction the Gaussian with covariance matrix $\Sigma = \frac{1}{2} \mathbf{C} \mathbf{C}^T$ is a Wigner function on the non-standard symplectic space (\mathbb{R}^4, ω) . However, a straightforward calculation reveals that it violates the standard RSUP (5).

3. Let us close our discussion with some clarifying remarks. The symplectic form ω such that $\mathcal{G}_{\Sigma, \zeta}$ is a quantum state on $(\mathbb{R}^{2n}; \omega)$ is not unique. Indeed, a phase-space function may be a Wigner function upon quantization on several distinct symplectic spaces (see [9] for examples). Moreover, although our result proves that a Gaussian will always be associated with some pure state on an appropriate symplectic space, it may nevertheless be a Wigner function on another symplectic space, this instance associated with a mixed state.

Finally, one could wonder whether any *reasonable* function on phase-space will always be a Wigner function on some appropriate symplectic space $(\mathbb{R}^{2n}; \omega)$. By a *reasonable* function, we mean a function which satisfies some obvious *a priori* requirements of Wigner functions: being real, normalized, uniformly continuous, bounded, square integrable, etc. The answer is negative. Indeed if a function $F(z)$ is a Wigner function on some $(\mathbb{R}^{2n}; \omega)$, then there has to exist a matrix \mathbf{S} such that $|\det \mathbf{S}| F(\mathbf{S}z)$ is a Wigner function on $(\mathbb{R}^{2n}; \sigma)$. But one may easily construct functions which are *reasonable* but are not Wigner functions on $(\mathbb{R}^{2n}; \sigma)$, even admitting rescalings of Planck's constant. The derivation of such examples is difficult and beyond the scope of the present work. It requires the concept of Narcowich-Wigner spectrum [15]. We shall get back to this issue in a future work.

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